DERIVACE SLOŽENÉ FUNKCE

$$[f(g(h(x)))]' = f'(g(h(x)))\cdot g'(h(x))\cdot h'(x)$$

– složenou funkci derivujeme tzv. po složkách

Př.: Derivujte funkci f(x) a výsledek upravte.

1) $f(x) = \ln(\sin x)$ → vnější složka: $y = \ln(u)$ → $y' = \frac{1}{u} = \frac{1}{\sin x}$
vnitřní složka: $u = \sin x$ → $u' = \cos x$

$$f'(x) = \underbrace{\frac{1}{\sin x}}_{y'}\cdot\underbrace{\cos x}_{u'} = \operatorname{cotg} x$$

2) $f(x) = \sin^2(3x-1)$
→ vnější složka: $y = u^2$ → $y' = 2u = 2\sin v = 2\sin(3x-1)$
→ střední složka: $u = \sin v$ → $u' = \cos v = \cos(3x-1)$
→ vnitřní složka: $v = 3x-1$ → $v' = 3$

$$f'(x) = \underbrace{2\sin(3x-1)}_{y'}\cdot\underbrace{\cos(3x-1)}_{u'}\cdot\underbrace{3}_{v'} = 3\cdot\sin[2(3x-1)] = 3\sin(6x-2)$$

$$\underbrace{2\sin\alpha\cos\alpha = \sin 2\alpha}_{(*)} \Rightarrow 2\cdot\sin(3x-1)\cdot\cos(3x-1) = \sin(6x-2)$$

V dalších příkladech již nebudeme složky rozepisovat, je označíme:

- červeně derivaci vnější složky
- zeleně derivaci vnitřní složky
- žlutě derivaci střední složky

(červeně a zeleně: u fcí skládajících se ze dvou složek; červeně, zeleně a žlutě: u fcí skládajících se ze tří složek)

3) $f(x) = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}\cdot(1-x^2)^{-\frac{1}{2}}\cdot(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

4) $f(x) = \ln \operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)$

$$f'(x) = \underbrace{\frac{1}{\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)}}_{=\operatorname{cotg}\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\frac{1}{2} = \frac{1}{2}\cdot\frac{\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)} =$$

$$= \underbrace{\frac{1}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cdot\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}}_{(*)} = \frac{1}{\sin\left(x+\frac{\pi}{2}\right)} = \frac{1}{\cos x}$$

5) $f(x)=\dfrac{1}{\ln^2(x^2+1)}=\ln^{-2}(x^2+1)$

$f'(x)=-2\cdot\ln^{-3}(x^2+1)\cdot\dfrac{1}{x^2+1}\cdot 2x=-\dfrac{4x}{(x^2+1)\cdot\ln^3(x^2+1)}$

6) $f(x)=\frac{1}{2}\ln^2\frac{1+x}{1-x}+tg^3(2x^2-5)$

$f'(x)=\frac{1}{2}\cdot 2\ln\frac{1+x}{1-x}\cdot\dfrac{1}{\frac{1+x}{1-x}}\cdot\dfrac{1\cdot(1-x)-(1+x)\cdot(-1)}{(1-x)^2}+3tg^2(2x^2-5)\cdot\dfrac{1}{\cos^2(2x^2-5)}\cdot 4x=$

derivace podílu fcí!

$=\dfrac{1-x+1+x}{(1+x)\cdot(1-x)}\cdot\ln\dfrac{1+x}{1-x}+12x\cdot tg^2(2x^2-5)\cdot\dfrac{1}{\cos^2(2x^2-5)}=$

$=\dfrac{2}{1-x^2}\cdot\ln\dfrac{1+x}{1-x}+\dfrac{12x\cdot tg^2(2x^2-5)}{\cos^2(2x^2-5)}$

7) $f(x)=x\cdot e^{-x^2}$ → součin fcí!

$f'(x)=1\cdot e^{-x^2}+x\cdot e^{-x^2}\cdot(-2x)=e^{-x^2}\cdot(1-2x^2)$

8) $f(x)=\dfrac{x^2}{(2-x)^2}$ → podíl fcí!

$f'(x)=\dfrac{2x(2-x)^2-x^2\cdot 2(2-x)\cdot(-1)}{(2-x)^4}=\dfrac{2x(2-x)(2-x+x)}{(2-x)^4}=\dfrac{4x}{(2-x)^3}$

9) $f(x)=\ln\dfrac{1-x^2}{1+x^2}$ → vnitřní složka je podíl fcí!

$f'(x)=\dfrac{1}{\frac{1-x^2}{1+x^2}}\cdot\dfrac{-2x\cdot(1+x^2)-(1-x^2)\cdot 2x}{(1+x^2)^2}=\dfrac{-2x(1+x^2+1-x^2)}{(1-x^2)\cdot(1+x^2)}=$

$=\dfrac{-4x}{1-x^4}=\dfrac{4x}{x^4-1}$

10) $f(x) = \operatorname{arctg} \frac{\sin x}{1+\cos x}$ → vnitřní složka je podíl fcí

$$f'(x) = \frac{1}{1+\left(\frac{\sin x}{1+\cos x}\right)^2} \cdot \frac{\cos x\,(1+\cos x) - \sin x \cdot (-\sin x)}{(1+\cos x)^2} =$$

$$= \frac{1}{\frac{(1+\cos x)^2+\sin^2 x}{(1+\cos x)^2}} \cdot \frac{\cos x + \overbrace{\cos^2 x + \sin^2 x}^{1}}{(1+\cos x)^2} = \frac{1+\cos x}{(1+\cos x)^2+\sin^2 x} =$$

$$= \frac{1+\cos x}{1+2\cos x+\underbrace{\cos^2 x+\sin^2 x}_{1}} = \frac{1+\cos x}{2+2\cos x} = \frac{1+\cos x}{2(1+\cos x)} = \frac{1}{2}$$

Samostatně derivujte:

11) $f(x) = \ln\left(x + \sqrt{x^2-2}\right)$

12) $f(x) = \operatorname{arctg} \frac{2+3x}{2-3x}$

13) $f(x) = 2\arcsin\sqrt{\frac{x}{2}} - \sqrt{2x-x^2}$

14) $f(x) = -\frac{\cos x}{2\sin^2 x} + \ln\sqrt{\frac{1+\cos x}{\sin x}}$

15) $f(x) = \cos^5(3x^2-x+2) \cdot \operatorname{cotg}\left(\frac{x}{4} - x^4\right)$

16) $f(x) = \frac{3e^{5+3x-2x^3}}{\sin^4\left(2-\frac{x}{3}\right)}$

11) $f(x) = \ln\left(x+\sqrt{x^2-2}\right) = \ln\left(x+(x^2-2)^{\frac{1}{2}}\right)$

$$f'(x) = \frac{1}{x+\sqrt{x^2-2}}\cdot\left(1+\frac{1}{2}(x^2-2)^{-\frac{1}{2}}\cdot 2x\right) = \frac{1}{x+\sqrt{x^2-2}}\cdot\left(1+\frac{x}{\sqrt{x^2-2}}\right) =$$

$$= \frac{1}{x+\sqrt{x^2-2}}\cdot\frac{\sqrt{x^2-2}+x}{\sqrt{x^2-2}} = \underline{\underline{\frac{1}{\sqrt{x^2-2}}}}$$

12) $f(x) = \operatorname{arctg}\frac{2+3x}{2-3x}$

$$f'(x) = \frac{1}{1+\left(\frac{2+3x}{2-3x}\right)^2}\cdot\frac{3\cdot(2-3x)-(2+3x)\cdot(-3)}{(2-3x)^2} =$$

$$= \frac{1}{\frac{(2-3x)^2+(2+3x)^2}{(2-3x)^2}}\cdot\frac{6-9x+6+9x}{(2-3x)^2} = \frac{12}{(2-3x)^2+(2+3x)^2} =$$

$$= \frac{12}{4-12x+9x^2+4+12x+9x^2} = \frac{12}{2(4+9x^2)} = \underline{\underline{\frac{6}{4+9x^2}}}$$

13) $f(x) = 2\arcsin\sqrt{\frac{x}{2}} - \sqrt{2x-x^2} = 2\arcsin\left(\frac{x}{2}\right)^{\frac{1}{2}} - (2x-x^2)^{\frac{1}{2}}$

$$f'(x) = 2\cdot\frac{1}{\sqrt{1-\left(\sqrt{\frac{x}{2}}\right)^2}}\cdot\frac{1}{2}\left(\frac{x}{2}\right)^{-\frac{1}{2}}\cdot\frac{1}{2} - \frac{1}{2}(2x-x^2)^{-\frac{1}{2}}\cdot(2-2x) =$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1-\frac{x}{2}}}\cdot\frac{1}{\sqrt{\frac{x}{2}}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2x-x^2}}\cdot(2-2x) = \frac{1}{2\sqrt{\frac{x}{2}\left(1-\frac{x}{2}\right)}} - \frac{2(1-x)}{2\sqrt{2x-x^2}} =$$

$$= \frac{1}{\sqrt{4\left(\frac{x}{2}-\frac{x^2}{4}\right)}} - \frac{1-x}{\sqrt{2x-x^2}} = \frac{1}{\sqrt{2x-x^2}} - \frac{1-x}{\sqrt{2x-x^2}} = \frac{1-1+x}{\sqrt{2x-x^2}} =$$

$$= \frac{x}{\sqrt{2x-x^2}} = \sqrt{\frac{x^2}{2x-x^2}} = \sqrt{\frac{x^2}{x(2-x)}} = \underline{\underline{\sqrt{\frac{x}{2-x}}}}$$

14) $f(x)=-\frac{\cos x}{2\sin^2 x}+\ln\sqrt{\frac{1+\cos x}{\sin x}}=-\frac{\cos x}{2\sin^2 x}+\ln\left(\frac{1+\cos x}{\sin x}\right)^{\frac{1}{2}}$

$$f'(x)=-\frac{-\sin x\cdot 2\sin^2 x-\cos x\cdot 2\cdot 2\cdot\sin x\cdot\cos x}{(2\sin^2 x)^2}+\frac{1}{\sqrt{\frac{1+\cos x}{\sin x}}}\cdot\frac{1}{2}\left(\frac{1+\cos x}{\sin x}\right)^{-\frac{1}{2}}\cdot\frac{-\sin x\cdot\sin x-(1+\cos x)\cdot\cos x}{\sin^2 x}=$$

$$=-\frac{-2\sin x(\sin^2 x+2\cos^2 x)}{4\sin^4 x}+\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1+\cos x}{\sin x}}}\cdot\frac{1}{\sqrt{\frac{1+\cos x}{\sin x}}}\cdot\frac{-(\sin^2 x+\cos x+\cos^2 x)}{\sin^2 x}=$$

$$=\frac{\sin^2 x+2\cos^2 x}{2\sin^3 x}-\frac{1}{2}\cdot\frac{1}{\frac{1+\cos x}{\sin x}}\cdot\frac{1+\cos x}{\sin^2 x}=\frac{\sin^2 x+2\cos^2 x}{2\sin^3 x}-\frac{1}{2\sin x}=$$

$$=\frac{\sin^2 x+2\cos^2 x-\sin^2 x}{2\sin^3 x}=\frac{2\cos^2 x}{2\sin^3 x}=\underline{\underline{\frac{\cos^2 x}{\sin^3 x}}}$$

15) $f(x)=\cos^5(3x^2-x+2)\cdot\cotg\left(\frac{x}{4}-x^4\right)$

$$f'(x)=5\cos^4(3x^2-x+2)\cdot[-\sin(3x^2-x+2)]\cdot(6x-1)\cdot\cotg\left(\tfrac{x}{4}-x^4\right)+$$
$$+\cos^5(3x^2-x+2)\cdot\left[-\frac{1}{\sin^2\left(\frac{x}{4}-x^4\right)}\right]\cdot\left(\tfrac{1}{4}-4x^3\right)=$$

$$=(5-30x)\cdot\sin(3x^2-x+2)\cdot\cos^4(3x^2-x+2)\cdot\cotg\left(\tfrac{x}{4}-x^4\right)+$$
$$+\underline{\underline{\frac{(16x^3-1)\cdot\cos^5(3x^2-x+2)}{4\sin^2\left(\frac{x}{4}-x^4\right)}}}$$

16) $f(x)=\frac{3e^{5+3x-2x^3}}{\sin^4\left(2-\frac{x}{3}\right)}$

$$f'(x)=\frac{3e^{5+3x-2x^3}\cdot(3-6x^2)\cdot\sin^4\left(2-\frac{x}{3}\right)-3e^{5+3x-2x^3}\cdot 4\sin^3\left(2-\frac{x}{3}\right)\cdot\cos\left(2-\frac{x}{3}\right)\cdot\left(-\frac{1}{3}\right)}{\left[\sin^4\left(2-\frac{x}{3}\right)\right]^2}=$$

$$=\frac{\left[3(3-6x^2)\sin\left(2-\frac{x}{3}\right)+4\cos\left(2-\frac{x}{3}\right)\right]\cdot\sin^3\left(2-\frac{x}{3}\right)\cdot e^{5+3x-2x^3}}{\sin^8\left(2-\frac{x}{3}\right)}=$$

$$=\underline{\underline{\frac{\left[(9-18x^2)\cdot\sin\left(2-\frac{x}{3}\right)+4\cos\left(2-\frac{x}{3}\right)\right]\cdot e^{5+3x-2x^3}}{\sin^5\left(2-\frac{x}{3}\right)}}}$$